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Geometrical Theories of Gravitation with Short-Range Forces

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Earlier geometrical theories of gravitation with short-range forces are analyzed, in view of a more general approach, to compare some of the properties of such theories with general relativity (GR). It is found that neither the scalar-tensor nor the fourth-order theories of gravity share with GR the interesting property that the binding energy of a gravitating system may be attributed to the loss of energy in packing the matter under its own gravitational field. This general approach, in the form of a GR field equation with an effective energy-momentum tensor is used to construct a constant-density, spherically-symmetric star model, via a heuristic argument, as a perturbation of the corresponding model in GR to study the modifications to the limiting gravitational mass. The application of the present study to other problems of physical interest is briefly mentioned.

1. INTRODUCTION

The theoretical possibilities of modifying the classical law of gravity at extremely large and extremely small distances have been raised in the past. For instance, Seeliger (1895) suggested that if the universe is not finite, then the Newtonian theory of gravitation should be modified at very large distances. Modifications of Einstein's theory of general relativity (GR) that incorporate a finite range of gravitation have also been discussed (Freund *et al.*, 1969; Boulware and Deser, 1972; Dehnen and Ghaboussi, 1987). Theories which attempt to unify gravity with the other forces of nature have usually led to predict deviations from Newton's inverse square law at given characteristic lengths scales, in the form of additional Yukawa terms [see Gibbons and Whiting (1981) for a comprehensive list of references]. Potentials of such modified form also occur in several scalar-tensor theories of gravity (O'Hanlon, 1972: Acharya and Hogan, 1973; Pimentel and Obregon,

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1986) and in higher derivative theories (Sexl and Pedana, 1966; de Witt, 1975; Stelle, 1978). Fischbach *et al.* (1986) suggested that there might be another fundamental force in nature which depends on the composition of the objects to add to the four existing forces (gravity, electromagnetism, and the strong and weak nuclear forces). Since then numerous experiments have been carried out to test whether there truly was a new force. It is now a general consensus, as a result of these experiments, that those original claims may have been mistaken, and that there is at present no firm evidence for a fifth force that depends on the composition of the objects (Gribbin, 1988). However, the research stimulated by those claims has produced strong evidence that there are deviations from Newton's law of gravity over ranges of a few hundred meters (Stacey *et al.*, 1987; Eckhardt *et al.*, 1988). These deviations affect all objects equally, and may include both a repulsive force and an attractive component. Although the actual need of additional short-range components of gravity will only be resolved by further experiments, if we wish to keep a geometrical description of gravitation, we have to consider seriously all those theories different from GR which incorporate in the weak-field, low-velocity limit such modifications. In spite of the fact that there exist many theories that satisfy this requirement, as we mention above, a systematic comparison of them with GR is still due.

The purpose of the present work is to provide a contribution in this direction by analyzing the scalar-tensor theories of the Kaluza–Klein–Jordan type[3] and the fourth-order theories of gravity. We wish to stress mainly those differences that originate in comparison with well-known properties of the Newtonian theory and are consequently inherited by GR in the weak-field, low-velocity limit. To this end, it will be enough to consider, in accordance with the geophysical measurements performed so far, the application of the theories in the weak-field limit to a bounded, static, spherically-symmetric distribution of matter.

The plan of the paper is as follows: In Section 2 we review briefly the family of scalar-tensor theories of the Kaluza–Klein–Jordan type and the fourth-order theories of gravity. In both cases we derive the weak-field static limit, and indicate how a solution to the field equations can be constructed in such a limit. In Section 3 we write the previous theories in the form of GR, i.e., with the Einstein tensor equal to an effective energy-momentum tensor constructed, in general, from the matter tensor, auxiliary fields, and the metric. This effective theory is used to compare a geometrical definition of binding energy (or mass defect) for a spherical system with the corresponding Newtonian definition, as the gravitational energy loss in forming the system from its constituents, initially at rest at infinity. We find that

[3]See Singh and Singh (1987) for a detailed review of a general class of scalar-tensor theories.

neither the scalar-tensor nor the fourth-order theories of gravity share with GR the interesting property that this mass defect may be attributed to the loss in energy in packing the matter under its own gravitational field. Of course, in the scalar-tensor theories it may be argued that this apparent disagreement may reside in that in forming the system we are also creating a scalar field which may itself support energy. However, a simple-minded calculation shows that even if we take into account a possible scalar field energy, the two ways of computing the binding energy produce different results. Our calculations are justified on the basis that in the geometrical theories of gravitation that we consider, the energy-momentum tensor satisfies locally a covariant conservation law, which in the weak-field limit reduces, as it is well known, to the Newtonian equation of motion for the fluid, consistently with the geodesic equation for a test particle. At the end of the section we take a perfect fluid of constant density, within a sphere of radius R, and integrate the equation for the pressure to study the possible modification introduced by a Yukawa correction to the limiting gravitational mass as given by GR.

We remark that it would be most desirable to extend the present study to a cosmological model (d'Olivo and Ryan, 1987) and also to construct physically interesting exact solutions, to check for possible deviations beyond the weak-field limit with respect to the predictions of GR. We present a summary and conclusions in Section 4.

2. THE SCALAR-TENSOR AND THE FOURTH-ORDER THEORIES OF GRAVITATION

In this section we describe briefly two geometrical theories of gravitation which lead in the slow-motion, weak-field limit to a Newtonian theory with additional Yukawa terms.

2.1. A Scalar-Tensor Theory of Gravity

The theory that we shall briefly present belongs to the family of scalar-tensor theories which are obtainable from a five-dimensional Lagrangian which after dimensional reduction produces the coupling between the scalar and tensor fields (Jordan, 1945; Kaluza, 1921; Klein, 1926; Bergmann, 1968). We begin by choosing the action in 4D spacetime for the scalar-tensor fields and matter as (Pimentel and Obregon, 1986)

$$I = \int [\phi R + \phi^{-1}\gamma(\phi) g^{ab}\phi_{,a}\phi_{,b} - 2\Omega(\phi) + 16\pi L_M]\sqrt{-g}\, d^4x \qquad (1)$$

where R is the scalar curvature; ϕ is the scalar field. The function $\Omega(\phi)$ is a potential for the scalar field and $\gamma(\phi)$ plays the role of a coupling function;

both γ and Ω are arbitrary functions of ϕ. The matter Lagrangian density is given by L_M. The field equations derived from (1) are

$$G_{ab} = -8\pi\phi^{-1}T_{ab} - \Omega\phi^{-1}g_{ab} - \phi^{-2}\gamma(\phi_{,a}\phi_{,b} - 1/2 g_{ab}g^{cd}\phi_{,c}\phi_{,d}) - \phi^{-1}(\phi_{;ab} - g_{ab}\Box_g\phi) \tag{2}$$

$$\Box_g\phi = -\frac{1}{2}\left(\frac{\gamma'}{\gamma} - \frac{1}{\phi}\right)g^{cd}\phi_{,c}\phi_{,d} + \frac{\phi}{2\gamma}(R - 2\Omega') \tag{3}$$

where T_{ab} is the covariant conserved energy-momentum tensor for matter; i.e., it satisfies the equation

$$T^{ab}{}_{;b} = 0 \tag{4}$$

as implied by the field equations (2) and (3). Thus, as in GR, a free neutral test particle will follow a geodesic line in the spacetime manifold. The scalar field ϕ influences the motion of a test particle only through the metric g_{ab}. By combining (2) and (3), the equation for ϕ can also be written in the form

$$\Box_g\phi = \frac{8\pi}{2\gamma+3}T - \frac{\gamma'}{2\gamma+3}g^{cd}\phi_{,c}\phi_{,d} + \frac{4\Omega}{2\gamma+3} - \frac{2\phi\Omega'}{2\gamma+3} \tag{5}$$

The Brans–Dicke theory is obtained when $\gamma(\phi) = \text{const}$ and $\Omega = 0$. General relativity corresponds to the limit $\gamma(\phi) \to \infty$ and $\Omega(\phi) \to \Lambda$ (the cosmological constant) and we consistently take $\phi \to \phi_0 + \bar{\phi}$, where $\phi_0 = \text{const}$ and $\bar{\phi} = O(\gamma^{-1})$. If we are interested in obtaining GR without the cosmological term, we shall take $\Omega(\phi_0) = 0$. Let us consider now the weak-field limit (without a cosmological constant) defined by the following assumptions:

$$\begin{aligned} T = O(1); \qquad \phi - \phi_0 &\equiv \bar{\phi} = O(1) \\ \Omega(\phi) &= \Omega_0'\bar{\phi} + O(2) \\ \Omega'(\phi) &= \Omega_0''\bar{\phi} + O(2) \\ \gamma(\phi) &= \gamma_0 + \gamma_0'\bar{\phi} + O(2) \\ \gamma'(\phi) &= \gamma_0' + \gamma_0''\bar{\phi} + O(2) \\ g_{ab} &= \eta_{ab} + h_{ab} + O(2) \end{aligned} \tag{6}$$

We substitute expressions (6) into (5). In the zeroth-order approximation we obtain $\Omega_0' = 0$, i.e., $\Omega(\phi) = O(2)$. The first-order equation is

$$\Box\bar{\phi} = \frac{8\pi}{2\gamma_0+3}T - \frac{2\phi_0\Omega_0''\bar{\phi}}{2\gamma_0+3} \tag{7}$$

where the box operator corresponds now to the flat metric and we have used the zeroth-order equation.

In the particular case of slow motion (or else for a static system) the lowest order equation that satisfies the field $\bar{\phi}$ is

$$\nabla^2\bar{\phi} - m^2\bar{\phi} = -8\pi\alpha\rho \tag{8}$$

where $m^2 = -2\phi_0\Omega_0''/(2\gamma_0+3)$; $\alpha = 1/(2\gamma_0+3)$, and we have taken $T = -\rho + O(2)$; $\rho = -T_4^4$, the mass density. The solution of (8) in R^3, which satisfies the boundary condition $\bar{\phi} \to 0$ at spatial infinity is

$$\bar{\phi}(\mathbf{x}) = 2\alpha \int_{R^3} G_Y(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\, d^3x' \tag{9}$$

where

$$G_Y(\mathbf{x}, \mathbf{x}') = \frac{\exp(-m|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} \tag{10}$$

is the whole-space Yukawa Green function; i.e., it satisfies the equation

$$\nabla^2 G_Y - m^2 G_Y = -4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{11}$$

and the boundary condition $G_Y = 0$ at infinity.

Let us consider now a static, spherically symmetric spacetime manifold $\mathcal{M}$ with metric

$$g = e^{\beta(r)}\, dr \otimes dr + r^2\, d\Omega^2 - e^{\eta(r)}\, dt \otimes dt$$

$$d\Omega^2 = d\theta \otimes d\theta + \sin^2\theta\, d\varphi \otimes d\varphi$$

Here r is an invariant quantity defined in such way that $1/r^2$ is the intrinsic Gaussian curvature of the (θ, φ) 2-space (intrinsically a sphere of radius r). We shall assume that β and η are bounded functions of r for $0 \leq r < \infty$ to be determined by the field equations. Since we are interested in the gravitational field of a bounded source, we shall further assume that $\mathcal{M}$ is asymptotically flat, i.e., β, $\eta \to 0$, at spatial infinity. The nonzero components of the mixed Einstein tensor are (Synge, 1960)

$$G_1^1 = \frac{1}{r^2} - e^{-\beta}\frac{1 + r\eta_1}{r^2} \tag{12}$$

$$G_2^2 = G_3^3 = e^{-\beta}\left(-\frac{\eta_{11}}{2} - \frac{\eta_1^2}{4} - \frac{\eta_1}{2r} + \frac{\beta_1}{2r} + \frac{\beta_1\eta_1}{4}\right) \tag{13}$$

$$G_4^4 = \frac{1}{r^2} - e^{-\beta}\frac{1 - r\beta_1}{r^2} \tag{14}$$

The subscript 1 on β and η means differentiation with respect to r; β and η can be solved quite easily in terms of G^1_1 and G^4_4. From (12), (14), and the boundary conditions on β and η we obtain

$$e^{-\beta}=1-\frac{1}{r}\int_0^r r'^2 G^4_4(r')\,dr' \tag{15}$$

$$\eta=\int_0^r\left(\frac{e^\beta-1}{r'}-r'e^\beta G^1_1\right)dr'+\text{const} \tag{16}$$

The constant in (16) is determined by the requirement $\eta\to 0$ at spatial infinity. The other components of G^a_b can be expressed in terms of G^1_1 and G^4_4 by making use of (13) or equivalently by the identity $G^{ab}{}_{;b}=0$.

In the weak-field approximation (12)-(14) take the form

$$G^4_4-G^1_1-G^2_2-G^3_3=\nabla^2\eta+O(2) \tag{17}$$

$$r^2G^1_1=\beta-r\eta_1+O(2) \tag{18}$$

$$r^2G^4_4=\beta+r\beta_1+O(2) \tag{19}$$

where we have assumed $O(\eta)=O(1)=O(\beta)$.

The motion of a neutral test particle in the weak-field limit will be governed by a Newtonian-like equation of motion with a gravitational potential V given by

$$V=\eta/2+O(2) \tag{20}$$

The first-order equations obtained from (2) are

$$\overset{1}{G}{}^a_b=-8\pi\phi_0^{-1}\overset{1}{T}{}^a_b-\phi_0^{-1}\overset{0}{g}{}^{ac}\bar\phi_{;cb}+\phi_0^{-1}\delta^a_b\Box\bar\phi \tag{21}$$

where $\overset{0}{g}{}^{ac}$ is the flat metric in spherical coordinates. Using (17)-(21), a straightforward calculation shows that

$$\nabla^2V=-4\pi\phi_0^{-1}\overset{1}{T}{}^4_4(1+\alpha)-m^2\phi_0^{-1}\bar\phi/2\equiv 4\pi\phi_0^{-1}\hat\rho \tag{22}$$

where we have defined $\hat\rho$ by the last identity. Since $\bar\phi$ is given by (9), we can express $\hat\rho$ in terms of ρ as

$$\hat\rho(x)=\rho(x)(1+\alpha)-\frac{\alpha m^2}{4\pi}\int G_Y(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')\,d^3x' \tag{23}$$

An interesting property of this (effective) mass distribution is that its integral over the whole space equals the total mass M of the system; i.e.,

$$M\equiv\int_{R^3}\rho\,d^3x=\int_{R^3}\hat\rho\,d^3x \tag{24}$$

To prove (24), we use definition (23) and (11). Notice that M, by (22), may also be considered as the total gravitational mass of the system as measured by a Keplerian observer at infinity.

Let us now find a solution to (22). To this end, define a new potential $V_N \equiv V + \bar{\phi}/2\phi_0$ and substitute $\bar{\phi}$ from (8) into (22) to obtain the expression

$$\nabla^2 V_N = 4\pi\phi_0^{-1}\rho \tag{25}$$

Thus, the function V_N is a Newtonian potential which is determined by the Poisson equation (25) and the boundary conditions. A solution to (22) may be expressed as

$$V = V_N + V_Y \tag{26}$$

where $V_Y \equiv -\bar{\phi}/2\phi_0$. The gravitational potential V contains a Newtonian part and a Yukawa correction whose range is determined by the constant Ω_0'' and a weighting factor α. The Newtonian limit is obtained when $\alpha \to 0$ ($|\gamma_0| \to \infty$) and $\phi_0 = G^{-1}$ (G is the gravitational constant). The parameters γ_0 and Ω_0'' are directly related to the experimental values of α and m (Stacey *et al.*, 1987; Eckhardt *et al.*, 1988).

The whole-space Green function corresponding to (22) is

$$V_1(r) = -G\left(\frac{1}{r} + \alpha\frac{e^{-mr}}{r}\right) \tag{27}$$

The potential V_1 may be considered as the gravitational potential (in the weak-field limit) of a unit mass at the origin.

2.2. Fourth-Order Theory of Gravity

We shall consider those models of classical gravity derived from actions that include both the Hilbert action and the four-derivative terms (i.e., terms quadratic in the curvature) (Stelle, 1978). It is well known that there are only two independent additions that one can possibly make, so that we have only a two-parameter family of field equations. We write the action in the form

$$I = \int (\lambda R_{ab}R^{ab} - \sigma R^2 + \kappa R + L_M)\sqrt{-g}\, d^4x \tag{28}$$

We obtain GR when λ and σ, considered to be dimensionless numbers, are equal to zero and $\kappa = (16\pi G)^{-1}$.

The field equations following from the action (28) are of the form

$$S_{ab} + \kappa G_{ab} = -\tfrac{1}{2} T_{ab} \tag{29}$$

where S_{ab} depends linearly on the parameters λ and σ, it contains second derivatives of the Ricci tensor and the scalar curvature, and quadratic terms in the curvature. Equations (29) are related by generalized Bianchi identities $(S^{ab} + \kappa G^{ab})_{;b} \equiv 0$, which are a direct consequence of the usual uncontracted Bianchi identities and the commutation relations. The matter energy-momentum tensor has to satisfy then the covariant conservation of the source, $T^{ab}{}_{;b} = 0$, so the equations of motion for a free particle are a consequence of the field equations, the same as in Einstein's theory.

In order to extract some of the physical consequences of (29), let us consider a static, spherically symmetric system in the weak-field approximation. We choose the metric in Schwarzschild coordinates with the same form as before, and work with the field equations up to terms linear in η or β. Again, there are only two independent equations, which we choose as the linear combinations

$$\overset{1}{S}{}_a{}^a + \kappa \overset{1}{G}{}_a{}^a = -1/2\,\overset{1}{T}{}_a{}^a \qquad \overset{1}{S}{}_i{}^i + \kappa \overset{1}{G}{}_i{}^i - \left(\overset{1}{S}{}_4{}^4 + \kappa \overset{1}{G}{}_4{}^4\right) = 1/2\,\overset{1}{T}{}_4{}^4$$

To simplify finding the general solution to these equations, we define the quantities

$$Y = r^{-2}\frac{d}{dr}(r\beta) \tag{30}$$

$$X = \nabla^2 V - Y \tag{31}$$

where V is given by (20). We obtain (Stelle, 1978)

$$4(6\sigma - 2\lambda)\nabla^2 X - 4\kappa X = \rho \tag{32}$$

$$2\lambda\nabla^2 Y - 2\kappa Y = \frac{\lambda - 4\sigma}{2(3\sigma - \lambda)}\rho + 2\kappa\frac{2\sigma - \lambda}{3\sigma - \lambda}X \tag{33}$$

To construct a solution, we first solve (32) to obtain X; we then replace this X into (33) and solve for Y; finally, with $X + Y$ as a source for V we solve the Poisson equation (31) to obtain the gravitational potential. Following Stelle (1978), we define the constants $m_0^2 = \kappa/(6\sigma - 2\lambda)$ and $m_2^2 = \kappa/\lambda$. A straightforward calculation shows that (32) and (33) can be written in the form

$$\nabla^2 X - m_0^2 X = \frac{m_0^2}{4\kappa}\rho \tag{34}$$

$$\nabla^2 Y - m_2^2 Y = -\frac{2m_2^2 + m_0^2}{6\kappa}\rho + \frac{2}{3}(m_2^2 - m_0^2)X \tag{35}$$

Let us consider now a bounded, spherically symmetric system in R^3. To obtain the whole-space Green functions associated with (31), (34), and (35) we take $\rho(\mathbf{x})=\delta(\mathbf{x}-\mathbf{x}')$. Let us denote those functions as V_1, X_1, Y_1 and impose the boundary condition

$$(V_1, X_1, Y_1)\to 0 \qquad \text{at spatial infinity} \tag{36}$$

We have to solve

$$\nabla^2 X_1 - m_0^2 X_1 = \frac{m_0^2}{4\kappa}\delta(\mathbf{x}-\mathbf{x}')$$

$$\nabla^2 Y_1 - m_2^2 Y_1 = -\frac{2m_2^2+m_0^2}{6\kappa}\delta(\mathbf{x}-\mathbf{x}')+\tfrac{2}{3}(m_2^2-m_0^2)X_1$$

$$\nabla^2 V_1 = X_1 + Y_1$$

with the boundary condition (36). We obtain the unique solution

$$X_1(\mathbf{x},\mathbf{x}') = -\frac{m_0^2}{16\pi\kappa}\frac{\exp(-m_0|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} \tag{37}$$

$$Y_1(\mathbf{x},\mathbf{x}') = \frac{m_2^2}{12\pi\kappa}\frac{\exp(-m_2|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} - \frac{m_0^2}{24\pi\kappa}\frac{\exp(-m_0|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} \tag{38}$$

$$V_1(\mathbf{x},\mathbf{x}') = \frac{1}{12\pi\kappa}\frac{\exp(-m_2|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{48\pi\kappa}\frac{\exp(-m_0|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{16\pi\kappa}\frac{1}{|\mathbf{x}-\mathbf{x}'|} \tag{39}$$

Since the system (31), (34), and (35) is linear in ρ, we have that the general solution which satisfies the boundary condition (36) is

$$V(r) = \int_{R^3} V_1(\mathbf{x},\mathbf{x}')\rho(r')\,d^3x' \tag{40}$$

$$Y(r) = \int_{R^3} Y_1(\mathbf{x},\mathbf{x}')\rho(r')\,d^3x' \tag{41}$$

The function β_1 to first order is obtained by integrating (30) with Y given by (38):

$$\beta_1(r) = \frac{1}{8\pi\kappa}\frac{1}{r} - \frac{1}{12\pi\kappa}\frac{e^{-m_2 r}}{r} - \frac{1}{24\pi\kappa}\frac{e^{-m_0 r}}{r} - \frac{m_2}{12\pi\kappa}e^{-m_2 r} - \frac{m_0}{24\pi\kappa}e^{-m_0 r} \tag{42}$$

Of course $\beta_1(0)=0$, as it should be according to (30). Notice from (39) that V_1 is the sum of the usual Newtonian potential plus two Yukawa potentials with ranges given by m_2 and m_0 in a fixed relative proportion, i.e., the coefficients do not depend on any free parameter and are just fixed by the coupling constant $\kappa = (16\pi G)^{-1}$ as required by the validity of the Newtonian limit at infinity. Moreover, at $r \equiv |\mathbf{x} - \mathbf{x}'| \to 0$ the $1/r$ term is canceled and (39) tends to the finite value $(48\pi\kappa)^{-1}(m_0 - 4m_2)$.

The peculiarities of the static field in the linear limit are better revealed if we consider an extended source. For this purpose we take a constant-density ($\rho = \rho_0$) spherical star model of radius R. The integrals in (40) and (41) are immediate, and we obtain (Chiang and Hamity, 1975)

$$Y(r) = \begin{cases} \dfrac{\rho_0}{3\kappa}\left[1 - e^{-m_2 R}\left(\dfrac{1}{m_2 r} + \dfrac{R}{r}\right)\sinh m_2 r\right] \\ \quad + \dfrac{\rho_0}{6\kappa}\left[1 - e^{-m_0 R}\left(\dfrac{1}{m_0 r} + \dfrac{R}{r}\right)\sinh m_0 r\right], & r \le R \\ \dfrac{\rho_0}{3\kappa}\dfrac{e^{-m_2 r}}{m_2 r}(m_2 R \cosh m_2 R - \sinh m_2 R) \\ \quad + \dfrac{\rho_0}{6\kappa}\dfrac{e^{-m_0 r}}{m_0 r}(m_0 R \cosh m_0 R - \sinh m_0 R), & r \ge R \end{cases} \tag{43}$$

$$V(r, R) = \begin{cases} -\dfrac{GM}{R}\left(\dfrac{3}{2} - \dfrac{r^2}{2R^2}\right) - \dfrac{GM}{m_0^2 R^3}\left[1 - \dfrac{e^{-m_0 R}}{m_0 r}(1 + m_0 R)\sinh m_0 r\right] \\ \quad + \dfrac{4GM}{m_2^2 R^3}\left[1 - \dfrac{e^{-m_2 R}}{m_2 r}(1 + m_2 R)\sinh m_2 r\right], & r \le R \\ -\dfrac{GM}{r} - \dfrac{GM}{m_0^3 R^3}\dfrac{e^{-m_0 r}}{r}(R m_0 \cosh m_0 R - \sinh m_0 R) \\ \quad + \dfrac{4GM}{m_2^3 R^3}\dfrac{e^{-m_2 r}}{r}(R m_2 \cosh m_2 R - \sinh m_2 R), & r \ge R \end{cases} \tag{44}$$

where we have made explicit the dependence of V on the size of the source; the integration constants are fixed by the condition that V is continuous at $r = R$; $M = \frac{4}{3}\pi R^3 \rho_0$. Notice that if we take the limit $R \to 0$, $\rho_0 \to \infty$, in (44), while keeping M constant, we obtain that the potential of a point particle equals MV_1, as expected. The potential outside the sphere of constant density deviates from the Newtonian form by Yukawa potentials with weighting factors depending on the size of the source and the range parameters m_0 and m_2.

Another interesting property, which is true in general and can be verified in our particular example using (43), is that $Y(r) \to 8\pi G\rho(r)$ in the limit of GR, $(m_0, m_2) \to \infty$.

3. THE EFFECTIVE ENERGY-MOMENTUM MODEL

From the form of the field equations (2) and (29) it is suggestive to consider those geometrical theories of gravitation with field equations

$$G_{ab} = -8\pi G\tilde{T}_{ab} \tag{45}$$

where the effective energy-momentum tensor $\tilde{T}_{ab}$ may contain explicitly the matter energy-momentum tensor, auxiliary fields, free parameters, and the metric components and their derivatives through geometrical tensors constructed from them. The tensor $\tilde{T}$ must satisfy the integrability conditions

$$\tilde{T}^{ab}{}_{;b} = 0 \tag{46}$$

as implied by the Bianchi identities.

The matter energy-momentum tensor will also be assumed to satisfy the covariant conservation laws of the source

$$T^{ab}{}_{;b} = 0 \tag{47}$$

which may be a consequence of the field equations and generalized Bianchi identities. So the equations of motion for a free particle will follow from (47) in the same way as in Einstein's theory. We shall also require that a well-defined limiting procedure exists such that (45) reduces to Einstein's equation in that limit.

We are particularly interested in theories which have as a slow-motion, weak-field limit a Newtonian potential with Yukawa corrections. More specifically, the gravitational potential should satisfy in the static linear limit the equation

$$\nabla^2 V = 4\pi G\hat{\rho} \tag{48}$$

with

$$V = V_N + \sum_j V_j \tag{49}$$

where the Newtonian potential V_N satisfies the Poisson equation

$$\nabla^2 V_N = 4\pi G\rho \tag{50}$$

and V_j are Yukawa potentials defined as solutions of the equations

$$\nabla^2 V_j - m_j^2 V_j = -4\pi G\alpha_j \rho(\mathbf{x}) \tag{51}$$

Thus, the effective mass density $\hat{\rho}$ for the case of a bounded distribution of matter and fields which go to zero at infinity is given by

$$\hat{\rho}(\mathbf{x}) = \int_{R^3} F_1(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\, d^3x' \tag{52}$$

where

$$F_1(\mathbf{x},\mathbf{x}') \equiv \left(1-\sum_j \alpha_j\right)\delta(\mathbf{x}-\mathbf{x}') + \frac{1}{4\pi}\sum_j \alpha_j m_j^2 \frac{\exp(-m_j|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} \tag{53}$$

(α_j, m_j) are constants. In the scalar-tensor theories of Section 2.1 we have only one set of parameters $\alpha_1 = -\alpha$ and $m_1 = m$ as defined in (8). For the fourth-order theory we have $(\alpha_1 = -1/3, m_1 = m_0)$ and $(\alpha_2 = 4/3, m_2 = m_2)$; clearly, $\alpha_1 + \alpha_2 = 1$; thus, the first term on the right-hand side of (53) is zero.

To make a comparison with GR of some properties of those theories for which a field equation of the form (45) is applicable, let us go back to a static, spherically symmetric spacetime manifold $\mathcal{M}$. The solution to the field equations, according to (15) and (16), can be formally expressed in the form

$$e^{-\beta(r)} = 1 + \frac{8\pi G}{r}\int_0^r r'^2 \tilde{T}_4^4(r')\,dr' \tag{54}$$

$$\eta(r) = -\beta(r) + 8\pi G\int_0^r r' e^{\beta}(\tilde{T}_1^1 - \tilde{T}_4^4)\,dr' + \text{const} \tag{55}$$

In GR, given $\tilde{T}_1^1 = T_1^1$ and $\tilde{T}_4^4 = T_4^4$ as functions of r, we determine the components of the metric through (54) and (55). The remaining components of T are given by (45) with G_a^b obtained from (12)-(14).

In the weak-field limit we shall obtain (20) and

$$\tilde{T}_4^4 - (\tilde{T}_1^1 + \tilde{T}_2^2 + \tilde{T}_3^3) = -\hat{\rho} + O(2) \tag{56}$$

Notice that we include terms like $\tilde{T}_1^1$ as being of the same order as $\tilde{T}_4^4 \sim O(\hat{\rho})$, in contrast to the weak-field limit in GR, where the stress components of T are of higher order. Actually, it is precisely an equation equivalent to (56) which was used before to obtain the functional form of $\hat{\rho}$ in both the scalar-tensor and higher-order theories.

For a static, spherically symmetric configuration, the total mean energy (rest mass + kinetic energy + compression energy + etc.) M_0 of the system can be expressed in the form (Misner *et al.*, 1971)

$$M_0 = -\int_{\Sigma} T^{ab} u_b\,d\sigma_a = \int_{R^3} \rho e^{\beta/2}\,d^3x \tag{57}$$

Here $\sum$ is a $t = \text{const}$ hypersurface. The product $e^{\beta/2}\,d^3x$ represents an invariant volume element, and $\rho = T_{u_a}^{ab} u_b$, $u^a = e^{-\eta/2}\delta_4^a$, is the energy density as measured by an observer with four-velocity u^a.

The total gravitational mass of the system seen at spatial infinity is defined by

$$M = \frac{1}{4\pi G}\lim_{r\to\infty}\int_{S_r^2} \text{grad } V \cdot d\sigma \tag{58}$$

where S_r^2 is a sphere of radius r with the surface-vector element $d\sigma$. To write (58), we have used the boundary condition $\eta \sim 2V \sim O(1/r)$. Equation (58) can be transformed into a volume integral over the source by using (48) and definition (52). Thus, we obtain

$$M = \int_{\text{Source}} \rho(r)\, d^3x \tag{59}$$

Equation (59) is precisely the same as in GR (or Newtonian theory) and can be proved in the same way as in the particular case of (24). The difference $E_G \equiv M - M_0$,

$$E_G = \int_{\text{Source}} \rho(1 - e^{\beta/2})\, d^3x \tag{60}$$

may be interpreted as the *binding gravitational energy of the system.*

In the nonrelativistic limit a binding gravitational energy is usually associated with the loss in energy in packing the matter under its own gravitational field, i.e., it is the work done by nongravitational forces against the gravitational field, in bringing the matter at rest at infinity to its final configuration. This *packing* energy is also the gravitational potential energy Ω of the system given by

$$\Omega = \frac{G}{2} \int V_1(|\mathbf{x} - \mathbf{x}'|)\, dm\, dm'$$

$$= \int_{\text{Source}} \tilde{V}(r)\rho(r)\, d^3x \tag{61}$$

The last formula in (61) has been particularized to spherical symmetry; $\tilde{V}(r)$ is the gravitational potential on the surface of a sphere of radius r:

$$\tilde{V}(r) = G \int_{R^3} V_1(|\mathbf{x} - \mathbf{x}'|)\rho(r')\theta(r - r')\, d^3x' \tag{62}$$

Here $r = |\mathbf{x}|$, $r' = |\mathbf{x}'|$, and $\theta(r - r')$ is the Heaviside step function. The potential $V_1(|\mathbf{x} - \mathbf{x}'|)$ is the solution of (48) for $\rho = \delta(|\mathbf{x} - \mathbf{x}'|)$ and boundary condition $V_1 \to 0$ as $|\mathbf{x} - \mathbf{x}'| \to \infty$. Outside the source we have $\tilde{V}(r) = V(r)$. Now, a natural question arises: Does E_G approach the value Ω in the weak-field limit? To answer this question, we take the weak-field limit of (60),

$$E_G \to \overset{1}{E}_G = -\frac{1}{2} \int_{\text{Source}} \rho \overset{1}{\beta}\, d^3x \tag{63}$$

The 1 on top of a given character indicates that we take that variable up to the first order $[\rho = O(1)]$. Comparing (61) and (63), we infer that a necessary and sufficient condition to have

$$\Omega = \overset{1}{E}_G$$

for any spherically symmetric distribution of energy ρ is that

$$2\tilde{V}(r) = -\overset{1}{\beta}(r) \tag{64}$$

In GR we have $\tilde{T}^a_b = T^a_b$; thus, (54) gives

$$\overset{1}{\beta}(r) = -2G\frac{m(r)}{r}$$

where $m(r)$ is the total mass enclosed by a sphere of radius r. On the other hand, we know that in Newtonian theory the potential $\tilde{V}(r) = -Gm(r)/r$. Therefore, we may conclude that in GR (64) is satisfied.

Let us consider now the scalar-tensor theories of Section 2. From (54) we can write the condition (64) in the equivalent form

$$\tilde{V}(r) = \frac{G}{r}\int_0^r \overset{1}{\tilde{T}}{}^4_4 \, d^3x' \tag{65}$$

Using (8), (9), (21), and (45), a straightforward calculation shows that

$$\overset{1}{\tilde{T}}{}^4_4 = -\rho + \alpha\rho - \frac{\alpha m^2}{4\pi}\int_{R^3} \frac{\exp(-m|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}')\, d^3x' \tag{66}$$

Let us consider again, for simplicity, the particular case $\rho(\mathbf{x}) = \rho_0\theta(R-r)$, $\rho_0 = \text{const}$. From (66) we obtain

$$\overset{1}{\tilde{T}}{}^4_4 = \begin{cases} -\rho + \alpha\rho - \alpha\rho_0\left[1 - \dfrac{e^{-mR}}{mr}(1+mR)\sinh mr\right], & r < R \quad (67) \\[2ex] -\rho + \alpha\rho - \alpha\rho_0 \dfrac{e^{-mr}}{mr}(mR\cosh mR - \sinh mR), & r > R \quad (68) \end{cases}$$

It is apparent from (67) that for $r < R$ the integral on the right-hand side in (65) depends on R, while $\tilde{V}(r)$ does not, according to definition (62). Thus, we can conclude that condition (64) will not be satisfied, in general, for any of the scalar-tensor theories of Section 2. A straightforward calculation will also show, in this particular case, that condition (64) is not satisfied for $r > R$ either. A similar reasoning applies to the fourth-order theories.

Although we have not investigated all the geometrical theories which in the slow-motion weak-field limit lead to the gravitational potential (48), along with (49)-(53), to determine whether condition (64) is fulfilled or not, we may assume that such a theory exists and study some of its properties. To this end, we shall construct a static spherically symmetric star model in the first order of approximation.[4] The validity of condition (64) allows us to dispose of one of the arbitrary functions which are required to define a unique solution to the field equations according to (54) and (55). In what follows we obtain that solution explicitly in terms of one arbitrary function $\rho(r)$.

Given $\rho(r)$, we obtain $\tilde{V}(r)$ from (62) and $\overset{1}{\tilde{T}}{}^4_4$ from (65), i.e.,

$$\overset{1}{\tilde{T}}{}^4_4 = \frac{1}{4\pi G r^2} \frac{d}{dr}[r\tilde{V}(r)] \tag{69}$$

From (56), (48), (45), (69), and the Bianchi identities, we obtain the following differential equation in first order:

$$3\left(\overset{1}{\tilde{T}}{}^1_1 - \overset{1}{\tilde{T}}{}^4_4\right) + r\frac{d}{dr}\left(\overset{1}{\tilde{T}}{}^1_1 - \overset{1}{\tilde{T}}{}^4_4\right) = \frac{1}{4\pi G}\nabla^2(V - \tilde{V}) \tag{70}$$

The solution of this equation is

$$\overset{1}{\tilde{T}}{}^1_1 - \overset{1}{\tilde{T}}{}^4_4 = \frac{1}{4\pi G r}\frac{dq}{dr} + \frac{C}{r^3} \tag{71}$$

where $q \equiv V(r) - \tilde{V}(r)$ and C is an integration constant which must be taken equal to zero to guarantee that the spacetime is locally Lorentz. In GR we have $\tilde{V}(r) = -Gm(r)/r$, $(dV/dr) = m(r)/r^2$, $m(r) = 4\pi\int_0^r T^4_4 r'^2\, dr'$; therefore, from (69) and (71) we obtain

$$\overset{1}{\tilde{T}}{}^4_4 = T^4_4; \qquad \overset{1}{\tilde{T}}{}^1_1 = 0 \tag{72}$$

as expected, since $\tilde{T}^1_1 = T^1_1 = O(2)$.

Outside the source ($r > R$) we have $V = \tilde{V}$, $\overset{1}{\tilde{T}}{}^1_1 = \overset{1}{\tilde{T}}{}^4_4$. Therefore, from (55) we obtain, in first order,

$$\overset{1}{\eta}(r) = -\overset{1}{\beta}(r) + \text{const}, \qquad r > R$$

which is a relationship between the metric coefficients similar to the one obtained in GR.

Let us consider now a perfect fluid

$$T^{ab} = \rho u^a u^b + p(g^{ab} + u^a u^b) \tag{73}$$

[4]Actually, condition (64) is just a relationship only valid in first order.

From (47) we obtain

$$\frac{dp}{dr}+\frac{p+\rho}{2}\frac{d\eta}{dr}=0, \qquad r\leq R \tag{74}$$

This last equation can be easily integrated for the case $\rho=\rho_0\theta(R-r)$. The solution which satisfies the condition $p(R)=0$ is

$$p(r)=\rho_0\{\exp\tfrac{1}{2}[\eta(R)-\eta(r)]-1\}, \qquad r\leq R \tag{75}$$

The pressure at the star center is given by

$$p_0=\rho_0\{\exp\tfrac{1}{2}[\eta(R,\rho_0)-\eta(0,\rho_0)]-1\} \tag{76}$$

where we have written the function η with an explicit ρ_0 dependency. We know that in GR, given a value of R, there exists a limiting value $\rho_0=\rho_c$ for which the central pressure becomes infinity. This value of ρ_0 imposes a limit to the total gravitational mass given by $(2GM/Rc^2)<8/9$. To study the possible modifications to this result in a theory with field equations (45), we may go a step further in our heuristic approach and add to (71) the GR value of $\tilde{T}^1_1$. More precisely, our aim is to write (74) up to an order of approximation in which the relativistic effects may show up. From (54) and (55) we have

$$\frac{d\eta}{dr}=2e^{\beta}\left[\frac{d\hat{V}}{dr}+4\pi Gr(\tilde{T}^1_1-\tilde{T}^4_4)\right] \tag{77}$$

where

$$\hat{V}=\frac{4\pi G}{r}\int_0^r \tilde{T}^4_4 r'^2\,dr'$$

It is clear that $\overset{1}{\hat{V}}(r)=\tilde{V}(r)$. In agreement with (71) and the GR limit, we make now the following assumption:

$$\tilde{T}^1_1=\overset{1}{\tilde{T}}{}^4_4+\frac{1}{4\pi Gr}\frac{d}{dr}(V-\tilde{V})+T^1_1+O(\alpha_j;2) \tag{78}$$

$$\tilde{T}^4_4=\overset{1}{\tilde{T}}{}^4_4+O(\alpha_j;2); \qquad \overset{1}{\tilde{T}}{}^4_4=T^4_4 \tag{79}$$

Here $O(\alpha_j;2)$ indicates terms which are of second order or higher, and are equal to zero when the parameters α_j vanish. From (77)-(79), we obtain

$$\frac{d\eta}{dr}=2e^{\beta}\left[\frac{dV}{dr}+4\pi Grp+O(\alpha_j;2)\right] \tag{80}$$

Thus, in the lowest relevant order, (74) becomes

$$\frac{dp}{dr} = -\frac{p+\rho}{1+2\tilde{V}}\left(\frac{dV}{dr} + 4\pi G r p\right) + \text{h.o.} \tag{81}$$

Equation (81) contains the exact GR limit ($\alpha_j \to 0$). From this point of view it may be considered as the first-order perturbation of the GR equation in the small parameters α_j.

As an example of the modifications with respect to GR introduced by our heuristic approach, we may compute the pressure p from (81) in the particular case $\rho = \rho_0 \theta(R-r)$. To this end, we neglect the h.o. terms and assume only one Yukawa potential characterized by (m, α) in (49). The expressions for $\tilde{V}$ and V can be easily obtained from (44) [take $m_0 = 0$, $m_2 = m$, and replace the factor 4 by α; recall that $\tilde{V}(r) \equiv V(r; r)$]. Equation (81) is then numerically integrated starting from the boundary condition $p(R) = 0$. We introduce dimensionless quantities by using geometrical units ($c = G = 1$) and defining new variables: $x = mr$, $X = mR$, $P = p/\rho_0$, $\rho_0' = \rho_0/m^2$. We choose $m^{-1} = 0.2$ km, $R = 10$ km, $\alpha = 10^{-3}$, $\rho_c' = \rho_c/m^2 \equiv (3\pi X^2)^{-1}$. In Figure 1 we show a qualitative comparison of the pressure in GR ($\alpha = 0$) with the corresponding result for $\alpha = 10^{-3}$. We notice that P is finite at $r = 0$. It can be seen that the pressure remains bounded if $\rho_0 \leq 1.0045\rho_c$, which

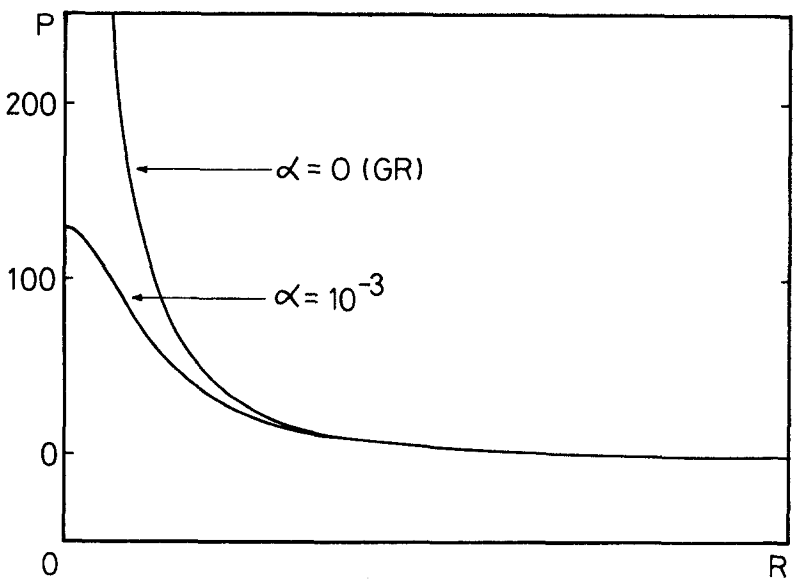

Fig. 1. A comparison of the radial pressure in a constant (critical) density model in GR ($\alpha = 0$) and the corresponding values with a repulsive Yukawa correction ($\alpha = 10^{-3}$). The pressure and radial distance are in relative units.

corresponds, in the present approximation, to a relative increase in the limiting gravitational mass of the same order as the constant α.

4. SUMMARY AND CONCLUSIONS

We conclude by summarizing the main features of the preceding approach, and by pointing out some interesting questions that remain to be discussed.

We have reviewed briefly two families of geometrical theories of gravitation which lead, in the weak-field, slow-velocity limit, to modifications of the Newtonian interaction incorporating short-range forces. This study suggested a possible general form to discuss any other theory with a nonrelativistic limit with such characteristics. Thus, we proposed field equations in the form of GR with an effective energy-momentum tensor constructed, in general, from the matter tensor, auxiliary fields, and the metric. This effective energy-momentum tensor must satisfy a covariant conservation law, as integrability conditions implied by the Bianchi identities. We have also assumed that the interaction between matter and geometry is in the usual form of a covariant conservation law for the matter energy-momentum tensor. This assumption implies that a neutral free particle will follow a geodesic line, just in the same way as in Einstein's theory. We used this result as a basis for a comparison of a geometrical definition of binding energy for a spherically symmetric static system in the weak-field limit with the corresponding Newtonian definition. We found that neither the scalar-tensor nor the fourth-order theories of gravity share with GR the interesting property that this mass defect may be attributed to the loss in energy in packing the matter under its own gravitational field.

Although we have not checked all possible theories which lead in the weak-field static limit to a Newtonian potential with Yukawa corrections to see whether the above condition is satisfied or not, we have assumed that such a theory exists, to construct, via a heuristic approach, a static, spherically symmetric star model in the lowest relevant order of approximation in an expansion in terms of the coupling constant of the Yukawa interaction. We found that the increase in the total gravitational mass that can be packed in the form of a sphere of constant density is of the same order as the coupling constant.

We remark that it would be most desirable to construct physically interesting exact solutions to the geometrical theories, to check for deviations beyond the weak-field limit with respect to predictions of GR. Finally, we believe that our study may be used to elucidate the dynamical equivalence between Einstein's theory and a class of nonlinear theories of gravitation (see, e.g., Ferraris *et al.*, 1988).

REFERENCES

Acharya, R., and Hogan, P. H. (1973). *Lettere al Nuovo Cimento*, **6**, 668

Bergmann, P. G. (1968). *International Journal of Theoretical Physics*, **1**, 25.

Boulware, D. G., and Deser, S. (1972). *Physical Review D*, **6**, 3368.

Chiang, C. C., and Hamity, V. H. (1975). *Nuovo Cimento* **30B**, 280.

Dehnen, H., and Ghaboussi, F. (1987). *International Journal of Theoretical Physics*, **26**, 483.

De Witt, B. (1975). *Physics Reports*, **19C**, 295.

D'Olivo, J. C., and Ryan, M. P. (1987). *Classical and Quantum Gravity*, **4**, L13.

Eckhardt, D., Jekeli, C., Lazarewicz, A. R., Romaides, A. J., and Sands, R. W. (1988). *Physical Review Letters*, **60**, 2567.

Ferraris, M., Francaviglia, M., and Magnano, O. (1988). *Classical and Quantum Gravity*, **5**, L95.

Fischbach, E., Sudarsky, D., Szafer, A., Talmadge, C., and Aronson, S. H. (1986). *Physical Review Letters*, **56**, 3.

Freund, P. G. O., Maheshwariu, A., and Schonberg, F. (1969). *Astrophysical J.*, **157**, 857.

Gibbons, G. W., and Whiting, B. F. (1981). *Nature*, **291**, 636.

Gribbin, J. (1988). *New Scientist*, **1988**, (25 February), 3.

Jordan, P. (1945). *Nachrichten der Akademie der Wissenschaften in Göttingen*, No. 39.

Kaluza, T. (1921). *Sitzungsberichte der Berliner Akademie der Wissenschaften*, No. 966.

Kim, J. K., and Yoom, J. (1987). *Physics Letters*, **121**, 213.

Klein, O. (1926). *Zeitschrift für Physik*, **37**, 895.

Misner, C. W., Thorne, K. S., and Wheeler, J. W. (1971). *Gravitation*, Freeman, San Francisco, Chapter 23.

O'Hanlon, J. (1972). *Physical Review Letters*, **29**, 137.

Pimentel, L. O., and Obregon, O. (1986). *Astrophysics and Space Science*, **126**, 231.

Seeliger, H. (1895). *Astronomischer Nachrichten*, **137**, (3273), 129.

Sexl, R., and Pedana, E. (1966). *Communications in Mathematical Physics*, **2**, 165.

Singh, T., and Singh, T. (1987). *International Journal of Modern Physics A*, **2**, 645.

Stacey, F. D., Tuck, G. J., Moore, G. I., Holding, S. C., Goodwin, B. D., and Zhou, R. (1987). *Review of Modern Physics*, **59**, 157.

Stelle, K. S. (1978). *General Relativity and Gravitation*, **9**, 353.

Synge, J. L. (1960). *Relativity, The General Theory*, North-Holland, Amsterdam.